



Some results on co-recursive associated Meixner and Charlier polynomials

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Abstract

An explicit representation of the co-recursive associated Meixner polynomials is given in terms of hypergeometric functions. This representation allows to derive a generating function, the Stieltjes transform of the orthogonality measure and the fourth-order difference equation verified by these polynomials. Special attention is given to some simple limiting cases occurring in the solution of the Chapman–Kolmogorov equation of linear birth and death processes. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The associated Meixner polynomials $\mathcal{M}_n(x; \beta, c, \gamma)$ have been studied in [8] where their Stieltjes function was derived. More recently the fourth-order difference equation they satisfy was given in [12]. The recurrence relation of the polynomials $\mathcal{M}_n(x; \beta, c, \gamma)$ is

$$\begin{aligned} c\mathcal{M}_{n+1}(x; \beta, c, \gamma) = & [(c-1)x + (c+1)(n+\gamma) + \beta c]\mathcal{M}_n(x; \beta, c, \gamma) \\ & - (n+\gamma)(n+\gamma+\beta-1)\mathcal{M}_{n-1}(x; \beta, c, \gamma), \end{aligned} \quad (1)$$

with the initial conditions:

$$\mathcal{M}_{-1}(x; \beta, c, \gamma) = 0, \quad \mathcal{M}_0(x; \beta, c, \gamma) = 1. \quad (2)$$

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The *real* association parameter is $\gamma > 0$. For $\gamma = 0$, Eq. (1) goes back to the recurrence relation of the Meixner polynomials $M_n(x; \beta, c)$ given, e.g., in [3, Eq. (3.2), p. 176].

We have the obvious relation

$$\mathcal{M}_n(x; \beta, c, 0) = M_n(x; \beta, c). \quad (3)$$

The co-recursive associated Meixner (CAM) polynomials $\mathcal{M}_n(x; \beta, c, \gamma, \mu)$ satisfy the same recurrence relation (1), with a real shift μ on the monic polynomial of first degree.

These polynomials are involved in the solution of the Chapman–Kolmogorov equation of the *most general* birth and death processes [6, 7, 9, 10] with linear transition rates

$$\lambda_n = c(n + \gamma + \beta), \quad \mu_{n+1} = n + 1 + \gamma, \quad n \geq 0; \quad \mu_0 = \gamma + (c - 1)\mu; \quad c \neq 1.$$

The case $\mu_0 = 0$, ($\mu = \gamma/(1 - c)$), correspond to the *honest* linear processes [14], (i.e., processes for which the sum of the probabilities $\sum_n p_{mn}(t) = 1$), and the corresponding particular CAM polynomials are the *zero-related* polynomials. Cases $\mu_0 = \text{Const.} \neq 0$ correspond to processes with absorption and are not *honest* [9]. However, if $\mu_0 = \gamma$, the corresponding processes are simply solved using the associated Meixner polynomials.

The aim of this article is to improve our understanding of the CAM polynomials $\mathcal{M}_n(x; \beta, c, \gamma, \mu)$ and derive the fourth-order difference equation, with respect to x , which they satisfy. In Section 2, we give two theorems. In Section 3, we construct an explicit form of these polynomials in terms of generalized hypergeometric series. In Section 4, we derive the Stieltjes transform of the orthogonality measure using a generating function of the CAM polynomials. In Section 5, we show that the explicit form derived in Section 3, implies the existence of a fourth-order difference equation. Using the symbolic computer algebra MAPLE [1], we give the coefficients of this equation. Finally, in Section 6, we derive the limiting form for the CAM polynomials in some particular cases, as well as for the co-recursive Meixner and the co-recursive associated Charlier polynomials.

2. Background material

In this section we give two theorems used in the following.

2.1. Contiguity relation ${}_3F_2\left(\begin{smallmatrix} a, b, e+1 \\ d, e \end{smallmatrix}; t\right)$

Theorem 1. The hypergeometric function ${}_3F_2\left(\begin{smallmatrix} a, b, e+1 \\ d, e \end{smallmatrix}; t\right)$ satisfies the contiguity relation

$$\begin{aligned} & (a - d - 1)[(b - e)(a - 1 - e)t - (e - d + 1)e] {}_3F_2\left(\begin{smallmatrix} a - 2, b, e + 1 \\ d, e \end{smallmatrix}; t\right) \\ & + [(b - e)(a - 2 - e)(a - b - 1)t^2 \\ & + ((-3a + d + 3 + b)e^2 + (2a[b - 3 + a] - 2db + 4)e - b(a - 2)(2a - d - 1))t \end{aligned}$$

$$\begin{aligned}
& + (2a - d - 2)(e - d + 1)e] {}_3F_2\left(\begin{matrix} a-1, b, e+1 \\ d, e \end{matrix}; t\right) \\
& + (1-t)(a-1)[(b-e)(a-2-e)t - (e-d+1)e] {}_3F_2\left(\begin{matrix} a, b, e+1 \\ d, e \end{matrix}; t\right) = 0.
\end{aligned} \tag{4}$$

Proof. A proof of this theorem can be obtained calculating the n th-order term in t . \square

Note that the hypergeometric function ${}_3F_2\left(\begin{matrix} a, b, e+1 \\ d, e \end{matrix}; t\right)$ can be expressed in terms of two ${}_2F_1$:

$$e {}_3F_2\left(\begin{matrix} a, b, e+1 \\ d, e \end{matrix}; t\right) = (e-a) {}_2F_1\left(\begin{matrix} a, b \\ d \end{matrix}; t\right) + a {}_2F_1\left(\begin{matrix} a+1, b+1 \\ d+1 \end{matrix}; t\right). \tag{5}$$

2.2. Fourth-order difference equation

In the following, we shall use the translation and identity operators

$$T: Tf_0(x) = f_0(x+1), \quad I: If_0(x) = f_0(x),$$

and for simplicity we often shall use the notation:

$$f_1(x) \equiv Tf_0(x) \quad \text{and more generally} \quad f_i(x) \equiv T^i f_0(x). \tag{6}$$

Theorem 2. If $y_0(x)$ and $z_0(x)$ satisfy the second-order difference equations

$$(T^2 + b_0T + c_0I)y_0(x) = 0, \quad (T^2 + B_0T + C_0I)z_0(x) = 0, \tag{7}$$

the product $u(x) \equiv u_0(x) = y_0(x)z_0(x)$ satisfies the fourth-order difference equation given in the determinantal form

$$\begin{vmatrix} L[u(x)] & B_0 & b_0 \\ M[u(x)] & -C_1b_1 & -c_1B_1 \\ N[u(x)] & c_2B_2C_1 & C_2b_2c_1 \end{vmatrix} = 0, \tag{8}$$

where

$$\begin{aligned}
L[u(x)] & \equiv u_2(x) - b_0B_0u_1(x) - c_0C_0u(x), \\
M[u(x)] & \equiv u_3(x) - b_1B_1u_2(x) - [c_1C_1 - c_1B_0B_1 - C_1b_0b_1]u_1(x), \\
N[u(x)] & \equiv u_4(x) - b_2B_2u_3(x) - [c_2C_2 - c_2B_1B_2 - C_2b_1b_2]u_2(x) \\
& \quad - [c_2B_2C_1b_0 + C_2b_2c_1B_0]u_1(x).
\end{aligned}$$

Proof. A proof of this theorem was given in [12] by iteration of Eq. (7). \square

The difference equation (8) will be written as

$$\{a_{(4)}(x)T^4 + a_{(3)}(x)T^3 + a_{(2)}(x)T^2 + a_{(1)}(x)T + a_{(0)}(x)I\}u(x) = 0. \tag{9}$$

3. Explicit form of the CAM polynomials

Let us first recall the relevant result from [12]. The recurrence relation (1) and the initial conditions (2) are invariant under the symmetry \mathcal{T} which maps the parameters (x, β, γ) into $(\bar{x}, \bar{\beta}, \bar{\gamma})$ according to

$$\mathcal{T} \begin{cases} \bar{x} = x + \beta - 1, & x = \bar{x} + \bar{\beta} - 1, \\ \bar{\beta} = 2 - \beta, & \beta = 2 - \bar{\beta}, \\ \bar{\gamma} = \gamma + \beta - 1, & \gamma = \bar{\gamma} + \bar{\beta} - 1, \end{cases} \quad \text{with } \mathcal{T}^2 \equiv \text{I}, \quad (10)$$

while the parameter c remains unchanged.

Theorem 3. *The explicit form of the CAM polynomials is*

$$M_n(x; \beta, c, \gamma, \mu) = (1 + \mathcal{T}) \frac{e}{1 - \beta} c^{x+\beta+\gamma-1} \frac{\Gamma(n + \gamma + 1)}{\Gamma(1 + \gamma)} \\ \times {}_3F_2 \left(\begin{matrix} -\gamma, -x, e + 1 \\ \beta, e \end{matrix}; \frac{c-1}{c} \right) {}_2F_1 \left(\begin{matrix} -n - \gamma - \beta + 1, -x - \beta + 1 \\ 2 - \beta \end{matrix}; \frac{c-1}{c} \right), \quad (11)$$

where $e = (1 - c)\mu - \gamma$. This form is valid under the positivity restrictions $\gamma > 0$, $\gamma + \beta > 0$, $\gamma + \beta \neq 1$, and $\beta \neq 1, 2, 3, \dots$

Eq. (11) displays explicitly the invariance under the transformation \mathcal{T} of the CAM polynomials.

Proof. Two linearly independent solutions of the recurrence relation (1) are [4, Eq. (31), p. 103]

$$u_n(x; \beta, c, \gamma) = \frac{\Gamma(n + \gamma + \beta)}{\Gamma(\beta)} {}_2F_1 \left(\begin{matrix} -n - \gamma, -x \\ \beta \end{matrix}; \frac{c-1}{c} \right) \quad (12)$$

and

$$\bar{u}_n(x; \beta, c, \gamma) \equiv \mathcal{T} u_n = \frac{\Gamma(n + \gamma + 1)}{\Gamma(2 - \beta)} {}_2F_1 \left(\begin{matrix} -n - \gamma - \beta + 1, -x - \beta + 1 \\ 2 - \beta \end{matrix}; \frac{c-1}{c} \right). \quad (13)$$

The value $\beta = 1$ has to be excluded since it is a fixed point of \mathcal{T} for which u_n and \bar{u}_n are obviously linearly dependent.

The Casorati determinant $W_n(u, \bar{u}) = u_n \bar{u}_{n-1} - \bar{u}_n u_{n-1}$ is

$$W_n(u, \bar{u}) = \frac{(\gamma)_n (\gamma + \beta - 1)_n}{c^n} W,$$

with $W \equiv W_0(u, \bar{u}) = u_0 \bar{u}_{-1} - u_{-1} \bar{u}_0$ given explicitly by

$$W = - \frac{\Gamma(\gamma) \Gamma(\gamma + \beta - 1)}{\Gamma(\beta) \Gamma(1 - \beta)} c^{1-\beta-\gamma-x}, \quad (14)$$

which is well defined for noninteger values of β and provided that $\gamma > 0$, $\gamma + \beta \neq 1$.

Using solutions (12) and (13) of the recurrence relation (1) we can write the CAM polynomials

$$M_n(x; \beta, c, \gamma, \mu) = K u_n(x; \beta, c, \gamma) + L \bar{u}_n(x; \beta, c, \gamma). \quad (15)$$

The associated Meixner polynomial of first degree is

$$\mathcal{M}_1(x; \beta, c, \gamma) = \frac{c-1}{c} \left[x + \frac{c+1}{c-1} \gamma + \frac{c}{c-1} \beta \right]. \quad (16)$$

The CAM polynomials being obtained by a shift μ on the monic polynomial of first degree their initial conditions can be written:

$$M_{-1}(x; \beta, c, \gamma, \mu) = \frac{\mu(1-c)}{\gamma(\gamma+\beta-1)}, \quad M_0(x; \beta, c, \gamma, \mu) = 1. \quad (17)$$

Note that these initial conditions are also invariant under the symmetry \mathcal{T} and let us observe that from Eq. (1) one has

$$M_n(x; \beta, c, \gamma, \mu) = c^{-n} M_n(-\beta - x; \beta, c^{-1}, \gamma, -\mu), \quad (18)$$

and therefore we can restrict our analysis to the range $0 < c < 1$.

Imposing the initial condition (17) we get readily

$$K = -\frac{1}{W} \left[\frac{\mu(1-c)}{\gamma(\gamma+\beta-1)} \bar{u}_0 - \bar{u}_{-1} \right], \quad L = \frac{1}{W} \left[\frac{\mu(1-c)}{\gamma(\gamma+\beta-1)} u_0 - u_{-1} \right], \quad (19)$$

and we can write, observing that $\bar{W} = \mathcal{T}W = -W$,

$$M_n(x; \beta, c, \gamma, \mu) = (1 + \mathcal{T}) \frac{U}{W} \bar{u}_n, \quad (20)$$

with

$$\frac{U}{W} = \frac{1}{W} \frac{\Gamma(\gamma+\beta-1)}{\Gamma(\beta)} \left[\frac{\mu(1-c)}{\gamma} {}_2F_1 \left(\begin{matrix} -\gamma, -x \\ \beta \end{matrix}; \frac{c-1}{c} \right) - {}_2F_1 \left(\begin{matrix} 1-\gamma, -x \\ \beta \end{matrix}; \frac{c-1}{c} \right) \right]. \quad (21)$$

Grouping the two ${}_2F_1$ using (5), and inserting (14) in (21), one gets

$$\frac{U}{W} = e^{x+\beta+\gamma-1} \frac{\Gamma(1-\beta)}{\Gamma(1+\gamma)} {}_2F_2 \left(\begin{matrix} -\gamma, -x, e+1 \\ \beta, e \end{matrix}; \frac{c-1}{c} \right), \quad \text{with } e = (1-c)\mu - \gamma. \quad (22)$$

Inserting (13) and (22) in Eq. (20) ends the proof of Theorem 3. \square

4. Stieltjes transform of the orthogonality measure

The associated Meixner polynomials were first studied in [8] where a generating function and the Stieltjes function were derived. Using the recurrence relation (1) and the initial condition (17) it is easy to prove that, if $\beta > 0$, $\gamma \geq 0$ and $c \neq 1$, one has for the CAM polynomials the following

generating function:

$$F(x, w) = \sum_{n \geq 0} \frac{(cw)^n}{(1 + \gamma)_n} \mathcal{M}_n(x; \beta, c, \gamma, \mu),$$

$$= (1 - w)^x (1 - wc)^{-\beta-x} \int_0^1 u^{\gamma-1} (1 - uw)^{-x-1} (1 - uwc)^{x+\beta-1} [\gamma - uw\mu(1 - c)] du. \quad (23)$$

Since the moment problem corresponding to these polynomials is always determined one can use Markov's theorem to get the Stieltjes transform of the orthogonality measure of the $\mathcal{M}_n(x; \beta, c, \gamma, \mu)$. Starting from the Stieltjes transform of the measure for the associated Meixner polynomials found in [8] and using the general result on the co-recursive polynomials obtained by Chihara [2, Section 4] we obtain the Stieltjes transform of the measure $d\Psi_\gamma^\mu$ for the CAM polynomials:

$$S_\gamma^\mu(z) = \int_0^\infty \frac{d\Psi_\gamma^\mu(s)}{z + s}$$

$$= \frac{{}_2F_1\left(\begin{matrix} 1 + \gamma, 1 - \beta + z \\ 1 + \gamma + z \end{matrix}; c\right)}{(z + \gamma){}_2F_1\left(\begin{matrix} \gamma, 1 - \beta + z \\ \gamma + z \end{matrix}; c\right) - \mu(1 - c){}_2F_1\left(\begin{matrix} \gamma + 1, 1 - \beta + z \\ \gamma + z + 1 \end{matrix}; c\right)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (24)$$

Regrouping the two ${}_2F_1$ in the denominator of (24) leads to the alternative form

$$S_\gamma^\mu(z) = \frac{1}{z + \gamma - \mu(1 - c)} \frac{{}_2F_1\left(\begin{matrix} 1 + \gamma, 1 - \beta + z \\ 1 + \gamma + z \end{matrix}; c\right)}{{}_3F_2\left(\begin{matrix} \gamma, 1 - \beta + z, g + 1 \\ \gamma + z + 1, g \end{matrix}; c\right)} \quad \text{with } g = \gamma \frac{\gamma + z - \mu(1 - c)}{\gamma - \mu(1 - c)}. \quad (25)$$

For $\mu = 0$ and $\mu = \gamma/(1 - c)$ we recover, respectively, the Stieltjes function of the associated Meixner polynomials $S_\gamma^a(z)$, and of the corresponding zero-related polynomials $S_\gamma^{zr}(z)$, given in [8].

$$S_\gamma^a(z) = \frac{1}{z + \gamma} \frac{{}_2F_1\left(\begin{matrix} 1 + \gamma, 1 - \beta + z \\ 1 + \gamma + z \end{matrix}; c\right)}{{}_2F_1\left(\begin{matrix} \gamma, 1 - \beta + z \\ \gamma + z \end{matrix}; c\right)}, \quad S_\gamma^{zr}(z) = \frac{1}{z} \frac{{}_2F_1\left(\begin{matrix} 1 + \gamma, 1 - \beta + z \\ 1 + \gamma + z \end{matrix}; c\right)}{{}_2F_1\left(\begin{matrix} \gamma, 1 - \beta + z \\ \gamma + z + 1 \end{matrix}; c\right)}. \quad (26)$$

5. Co-recursive associated Meixner difference equation

Theorem 4. *The co-recursive associated Meixner polynomials $\mathcal{M}_n(x; \beta, c, \gamma, \mu)$ satisfy the difference equation*

$$\{a_{(4)}(x)T^4 + a_{(3)}(x)T^3 + a_{(2)}(x)T^2 + a_{(1)}(x)T + a_{(0)}(x)I\}u(x) = 0, \quad (27)$$

with

$$\begin{aligned}
 a_{(4)}(x) &= f_1 f_2 F_1 F_2 (d_0 d_1 E_1 F_0 - e_1 f_0 D_0 D_1), \\
 a_{(3)}(x) &= f_1 F_1 [d_0 D_2 E_1 F_0 (e_2 f_1 - d_1 d_2) + d_2 e_1 f_0 D_0 (D_1 D_2 + E_2 F_1)], \\
 a_{(2)}(x) &= d_1 E_1 E_2 F_0 F_1 (d_0 d_1 d_2 - d_2 e_1 f_0 - d_0 e_2 f_1) \\
 &\quad + e_1 e_2 f_0 f_1 D_1 (D_2 E_1 F_0 + D_0 E_2 F_1 - D_0 D_1 D_2), \\
 a_{(1)}(x) &= -e_1 E_1 (-d_2 e_1 f_0 D_0 E_2 F_1 + d_0 e_2 f_1 D_2 E_1 F_0 + d_0 d_1 d_2 D_0 E_2 F_1 - d_1 e_2 f_1 D_0 D_1 D_2), \\
 a_{(0)}(x) &= e_0 e_1 E_0 E_1 (d_1 d_2 E_2 F_1 - e_2 f_1 D_1 D_2),
 \end{aligned} \tag{28}$$

where we use notation (6) for the functions of x , defined in Eqs. (30), (34–36), and used in the right-hand sides of Eqs. (28).

Proof. In view of using Theorem 2, we have to find the second order difference equations satisfied by each term of the product in the representation (11) of the CAM polynomials.

Difference equation for \bar{u}_n . Using the contiguity relation [4, Eq. (31), p. 103] we know that \bar{u}_n , (Eq. 13), is the solution of the difference equation

$$\{T^2 + b_0(x)T + c_0(x)I\} u_n = 0, \tag{29}$$

with

$$\begin{cases} b_0(x) = \frac{d_0(x)}{f_0(x)}, & d_0(x) = N + R, \\ & e_0(x) = x + \beta, \\ c_0(x) = \frac{e_0(x)}{f_0(x)}, & f_0(x) = c(x + 2), \end{cases} \tag{30}$$

where

$$R(x) = \gamma - x - 2 - c(\gamma + \beta + x) \quad \text{and} \quad N = (n + 1)(1 - c). \tag{31}$$

Difference equation for U/W . In view of Theorem 1 it is easy to prove that U/W given by (22) is the solution of the following difference equation:

$$\{T^2 + B_0(x)T + C_0(x)I\} \frac{U}{W} = 0, \tag{32}$$

with

$$B_0(x) = \frac{D_0(x)}{F_0(x)}, \quad C_0(x) = \frac{E_0(x)}{F_0(x)}, \tag{33}$$

where

$$\begin{aligned}
 D_0(x) &= (\beta + \gamma + x)[(x + 2 + e)\gamma + (x + \beta + 1)e]c^2 \\
 &\quad - [e(e + 2\gamma)(\gamma + x + \beta + 1) + \gamma(2\gamma - 1 + \beta)(x + 2)]c - (\gamma + e)(x + 2 + e)(x - \gamma + 1),
 \end{aligned} \tag{34}$$

$$E_0(x) = c(1+x)[(\gamma+e)(x+2+e) - ((x+2+e)\gamma + (x+\beta+1)e)c], \quad (35)$$

$$F_0(x) = (x+\beta+1)[(x+1+e)(\gamma+e) - ((x+1+e)\gamma + (x+\beta)e)c]. \quad (36)$$

Difference equation for $U/W \times \bar{u}_n$. The functions U/W and \bar{u}_n being both solution of second-order difference equations the product $U/W \bar{u}_n$ satisfies, by Theorem 2, a fourth-order difference equation. To obtain, from Eq. (8), the coefficients $\{a_{(i)}(x), i=0, \dots, 4\}$, given in Eq. (28), we have used the symbolic computer algebra MAPLE [1].

Difference equation for $\mathcal{M}_n(x; \beta, c, \gamma, \mu)$. To prove that the CAM polynomials $\mathcal{M}_n(x; \beta, c, \gamma, \mu)$ satisfy a fourth-order difference equation it remains to show that $\mathcal{T}U/W \bar{u}_n$ is also a solution of the *same* difference equation. This is a tricky point which follows from the observation that the hypergeometric functions

$$u_n, \quad \frac{\Gamma(x+1)\Gamma(\beta)}{\Gamma(x+\beta)} \bar{u}_n,$$

are solutions of the difference equation (29) while

$$\frac{U}{W}, \quad \frac{\Gamma(x+\beta)\Gamma(2-\beta)}{\Gamma(x+1)} \frac{\bar{U}}{\bar{W}},$$

are solutions of the difference equation (32). So the coefficients of the fourth-order difference equation satisfied by the CAM polynomials are given by Eqs. (28). \square

6. Particular cases

We now give different results corresponding to limiting cases of special interest in the solution of the Chapman–Kolmogorov equation of linear birth and death processes.

6.1. Particular CAM polynomials

In the limit $\mu=0$ we obtain the associated Meixner polynomials studied in [12]. There are two other simple cases of associated polynomials for which the Stieltjes functions are also ratios of two ${}_2F_1$.

6.1.1. Limit $\mu = \gamma/(1-c)$

In this limit we obtain the so-called *zero-related* Meixner polynomials studied in [8]. We note that the symmetry \mathcal{T} is now broken. The coefficients of the difference equation are:

$$a_{(4)}(x) = c(x+3)(x+\beta+3)[N(N+2R-c-1)(x+1) - R(R+2x-c+3) + c(x+2)],$$

$$\begin{aligned} a_{(3)}(x) = & -N[N(x+1)(R-2c-1)(N+3R-3c-3) + (2x+1)R^3 - 3((1+c)(3x+2)+1)R^2 \\ & + (c(2c(6x+5) + 25x + \beta + 28) + 10 + 9x)R + 2x(c(1+c)x^2 + c(1+c)(\beta+6)x \\ & - 2c^3 + (5+4\beta)c^2 + (7+4\beta)c - 1) - 4c^3 + (-2+6\beta)c^2 + (7\beta+4)c - 2] \end{aligned}$$

$$\begin{aligned}
& + R^4 + (2x - 5c)R^3 - (c(10x - 8c + \beta + 7) + 6x + 7)R^2 \\
& + (c(2c(7x - 2c + \beta + 9) + 16x + \beta + 24) + 4x + 6)R \\
& + 2c(1 + c)(x + 2)(x(x + \beta + 4) - 2c + 2\beta + 3), \\
a_{(2)}(x) = & -(x + 1)N[-N^3 + 4(-R + c + 1)N^2 + (-6R^2 + 12(1 + c)R + 2xc(x + \beta + 4) - 5c^2 \\
& + c(5\beta - 2) - 5)N - 4R^3 + 12(1 + c)R^2 + 2(2xc(x + \beta + 4) - 5c^2 + c(5\beta - 2) - 5)R \\
& - (1 + c)(2xc(2x + 2\beta + 7) - 2c^2 + (8 + 9\beta)c - 2)] - 2R^4 - 4(x - 2c + 1)R^3 \\
& + [x((2x + 2\beta + 19)c + 6) - 10c^2 + (4\beta + 23)c + 8]R^2 + [x(4xc(x - c + \beta + 5) \\
& - (4\beta + 21)c^2 + (14\beta + 27)c - 2) + 4c^3 - (6\beta + 27)c^2 + (12\beta + 9)c - 2]R \\
& - c[x(2x(-c^2 + (2x + 9 + 2\beta)c + x + \beta + 4) - (2\beta + 6)c^2 + (13\beta + 28)c \\
& + 6\beta + 10) - 2(4 + \beta)c^2 + (11\beta + 14)c + 4(1 + \beta)], \\
a_{(1)}(x) = & -N[N(x + 2)(R + 1)(N + 3(R - c - 3)) + (2x + 3)R^3 - 3((c + 1)(x + 1) + 1)R^2 \\
& - (c(x + 2c + \beta - 1) + 3x + 5)R - 2x(c + 1)(c(x^2 + (5 + \beta)x + 4\beta + 7) - 1) \\
& - (8 + 6\beta)c^2 - (2 + 7\beta)c + 4] + R^4 + (2x - 3c + 4)R^3 - (c(2x - 2c + 7 - \beta) + 1)R^2 \\
& - (2(x + \beta)c^2 + (4x + \beta + 6)c + 2x + 4)R - 2c(c + 1)(x + \beta + 1)(x + 2)^2, \\
a_{(0)}(x) = & c(x + 3)(x + \beta + 3)[(x + 2)N(N + 2R - 3c - 3) \\
& - R(R + 2x - 3c + 3) - 2c^2 + (3x + 5)c + 2(x + 2)],
\end{aligned}$$

where R and N are defined in (31).

6.1.2. Limit $\mu = (\gamma + \beta - 1)/(1 - c)$

This is another simple case of co-recursive associated Meixner polynomials for which the coefficients of the difference equation are obtained from those of the *zero-related* case in Section 6.1.1 by the transformation \mathcal{T} given by Eqs. (10). Note that R and N defined in (31) are invariant in the symmetry \mathcal{T} .

6.2. Co-recursive Meixner polynomials

The co-recursive Meixner polynomials are obtained in the limit $\gamma = 0$. In a similar way as the fourth-order differential equations satisfied by the co-recursive Laguerre or Jacobi polynomials factorize [11, 15], the fourth-order difference equation (27) satisfied by these polynomials can be written into a (2×2) difference equation of the form

$$\begin{aligned}
& [A(x)T^2 + B(x)T + C(x)I] \\
& \times [(x + 2)T^2 + ((n + 1)(1 - c) - x - 2 - c(x + \beta))T + c(x + \beta)I] \mathcal{M}_n(x; \beta, c, 0, \mu) = 0, \quad (37)
\end{aligned}$$

where the coefficients $A(x)$, $B(x)$ and $C(x)$ computed with MAPLE are given by

$$\begin{aligned}
 A(x) = & -c(\beta + 3 + x)[n(c(\beta + 2 + x) - x - 3)(c(x + \beta) - 1 - x)((c - 1)n + (2x + 3)(c + 1) \\
 & + 2c\beta) + c(\beta + 2 + x)((\beta + 1 + x)(x + \beta)c^2 - (1 + x)(\beta + 2 + x)c - (1 + x)^2) \\
 & + (3 + x)(2 + x)(1 + x) + \mu(c - 1)(n((2 + 2x + 2\beta + \mu)c - 4 - 2x - \mu)((-1 + c)n \\
 & + (2x + 3)(c + 1) + 2c\beta) + (\beta + 2 + x)(\beta + 1 + x)c^2 - (2 + x)(1 + x))], \\
 B(x) = & n[n((c - 1)n + (3x + 6)(c + 1) + 3c\beta)(c - 1)((\beta + 3 + x)c - 4 - x)(c(x + \beta) - 1 - x) \\
 & + (x + \beta)(\beta + 3 + x)(2x^2 + 4(\beta + 2)x + 2(\beta + 2)^2 - 1)c^4 - (4x^3 + 3(7 + 4\beta)x^2 \\
 & + 3(11\beta + 12 + 4\beta^2)x + 4\beta^3 + 21 + 15\beta + 12\beta^2)c^3 - (4x^4 + 8(\beta + 4)x^3 + (84 + 46\beta \\
 & + 4\beta^2)x^2 + (14\beta^2 + 76\beta + 80)x + 17\beta^2 + 33\beta + 22)c^2 + (27x^2 + 4x^3 + (60 - 9\beta)x + 43 \\
 & - 15\beta)c + (x + 4)(1 + x)(2x^2 + 8x + 7)] + c(\beta + 3 + x)(c^2(\beta + 2 + x)((\beta + 1 + x) \\
 & \times (x + \beta)c - 1 - x) - (1 + x)((2x^2 + 2(\beta + 4)x + 3\beta + 8)c - 2 - x)) \\
 & + (x + 4)(3 + x)(2 + x)(1 + x) + \mu(c - 1)[n(n(c - 1)((c - 1)n + (3x + 6)(c + 1) \\
 & + 3c\beta)(c(\mu + 2x + 3 + 2\beta) - \mu - 5 - 2x) + ((2x + 2\beta + 5)(\beta + 1 + x)c^3 + (x(2x + 9) \\
 & + 9 - 2\beta^2)c^2 - (x(2x + 7 + 4\beta) + 5 + 7\beta)c - (3 + x)(2x + 3))\mu + (4x^3 + 3(7 + 4\beta)x^2 \\
 & + 6(2\beta^2 + 7\beta + 6)x + 4\beta^3 + 21\beta^2 + 36\beta + 21)c^3 + (4x^3 + (8\beta + 19)x^2 \\
 & + (22\beta + 26 + 4\beta^2)x + 9\beta + 11 + 3\beta^2)c^2 - (4x^3 + 45 + 17\beta + (29 + 4\beta)x^2 \\
 & + (66 + 16\beta)x)c - 4x^3 - 27x^2 - 60x - 43) + c(\beta + 3 + x)((\beta + 2 + x)(\beta + 1 + x)c^2 \\
 & + (1 + x)(\beta + 2 + x)c - (2 + x)(1 + x)) - (3 + x)(2 + x)(1 + x)], \\
 C(x) = & -(1 + x)[n((-1 + c)n + (5 + 2x)(c + 1) + 2c\beta)((\beta + 3 + x)c - 4 - x)(c(\beta + 1 + x) \\
 & - 2 - x) + c(\beta + 3 + x)((\beta + 2 + x)(\beta + 1 + x)c^2 - (2 + x)(\beta + 3 + x)c - (2 + x)^2) \\
 & + (x + 4)(3 + x)(2 + x) + \mu(-1 + c)(n((-1 + c)n + (5 + 2x)(c + 1) + 2c\beta)((4 + 2x \\
 & + 2\beta + \mu)c - 6 - 2x - \mu) + (\beta + 3 + x)(\beta + 2 + x)c^2 - (3 + x)(2 + x))].
 \end{aligned}$$

6.3. Co-recursive associated Laguerre polynomials

The co-recursive associated Laguerre polynomials can be reached from the CAM polynomials through the limiting procedure

$$\beta = \alpha + 1, \quad x = z(1 - c)^{-1}, \quad \mu = \mu(1 - c)^{-1}, \quad c \rightarrow 1,$$

according to

$$\lim_{c \rightarrow 1} M_n \left(\frac{z}{1-c}; \alpha + 1, c, \gamma, \frac{\mu}{1-c} \right) = (1 + \gamma)_n \mathcal{L}_n^{(\alpha)}(z, \gamma, \mu). \quad (38)$$

In this limit the difference operator T becomes a differential operator with

$$T = 1 + (1 - c)D_z, \quad \text{where } D_z = \frac{d}{dz},$$

and we indeed recover the results obtained for the co-recursive associated Laguerre in [11, Eq. (2.39)].

6.4. Co-recursive associated Charlier polynomials

The co-recursive associated Charlier polynomials $\mathcal{C}_n(x; a, \gamma, \mu)$ have for recurrence relation (see, e.g., [5, Eq. 8, p. 227])

$$a\mathcal{C}_{n+1}(x; a, \gamma, \mu) = (n + \gamma + a - x)\mathcal{C}_n(x; a, \gamma, \mu) - (n + \gamma)\mathcal{C}_{n-1}(x; a, \gamma, \mu), \quad n \geq 0, \quad (39)$$

with the initial conditions:

$$\mathcal{C}_{-1}(x; a, \gamma, \mu) = \frac{\mu}{\gamma}, \quad \mathcal{C}_0(x; a, \gamma, \mu) = 1. \quad (40)$$

They can be reached from the CAM polynomials through the limiting procedure [12]

$$\lim_{\beta \rightarrow \infty} \frac{1}{(\gamma + \beta)_n} M_n \left(x; \beta, \frac{a}{a + \beta}, \gamma, \mu \right) = \mathcal{C}_n(x; a, \gamma, \mu). \quad (41)$$

The coefficients of the difference equation satisfied by them are obtained by applying this limit on the difference equation of the CAM polynomials.

In the limit $\mu = 0$ we recover the difference equation [12, Eq. (40)] for the associated Charlier polynomials.

6.4.1. Zero-related Charlier polynomials

The coefficients of the difference equation for the *zero-related* Charlier polynomials are obtained when $\mu = \gamma$,

$$\begin{aligned} a_{(4)}(x) &= a(3 + x)[n(1 + x)(n + 2R + 1) - R(R + 1)], \\ a_{(3)}(x) &= n[n(1 + x)(R - 1)(n + 3R) + (1 + 2x)R^3 - (3 + 3x)R^2 + (1 + a)R + 2ax^2 \\ &\quad + (1 + 8a)x + 1 + 7a] - R^4 + R^3 + (1 + a)R^2 - R - a + 36, \\ a_{(2)}(x) &= n(1 + x)[n(n(n + 4R) + 6R^2 - (5 + 2x)a - 1) + 4R^3 - ((4x + 10)a + 2)R - a] \\ &\quad + 2R(-R^3 + (1 + a(x + 2))R + a), \\ a_{(1)}(x) &= n[n(x + 2)(R + 1)(n + 3R) + (2x + 3)R^3 + (6 + 3x)R^2 + (1 - a)R \\ &\quad - x(2a(x + 4) + 1) - 2 - 7a] - R^4 - R^3 - (-1 + a)R^2 + R + 12 + a, \\ a_{(0)}(x) &= a(1 + x)[n(x + 2)(n + 2R - 1) - R(R - 1)], \end{aligned}$$

where $R = \gamma - x - 2 - a$. Because of the lack of symmetry in the recurrence relation (39) there is no other simple case of co-recursive associated Charlier polynomials.

6.4.2. Co-recursive Charlier polynomials

The co-recursive Charlier polynomials are obtained in the limit $\gamma = 0$. The fourth-order difference equation they satisfy can be factorized into a (2+2) difference equation of the form

$$[A(x)T^2 + B(x)T + C(x)I][(x+2)T^2 + (n+R+1)T + aI]\mathcal{C}_n(x; \beta, c, 0, \mu) = 0, \quad (42)$$

where the coefficients $A(x)$, $B(x)$ and $C(x)$ are given by:

$$\begin{aligned} A(x) &= a[n(R+2a-1)(R+2a+1)(n+2R+1) + R^3 + 4aR^2 + (4a^2 + 2a - 1)R + a^2 \\ &\quad + \mu(n(n+2R+1)\mu - 2n(R+2a)(n+2R+1) - R^2 - (1+2a)R - a)], \\ B(x) &= n[n(R+2a-2)(R+2a+1)(n+3R) + 2R^4 + (8a-2)R^3 + (1+2a)(4a-5)R^2 \\ &\quad + (1-a-8a^2)R + 2 + 4a - a^2 - 8a^3] \\ &\quad + R^4 + (4a-2)R^3 - (1+5a-4a^2)R^2 + (2-a-7a^2)R + 2a - a^2 - 4a^3 \\ &\quad + \mu[n\mu(-n(-n-3R) + 2R^2 + 1 + x - a) + n(-n(-2R-4a+1)(-n-3R) \\ &\quad - 4R^3 - (8a-3)R^2 + 6aR - 1 + a + 8a^2) - R^3 - 2aR^2 + (1+a)R + a(1+2a)], \\ C(x) &= (1+x)[n(R+2a-2)(R+2a)(n+2R-1) + R^3 + (4a-3)R^2 + (2-6a+4a^2)R \\ &\quad + a(2-3a) + \mu(n(n+2R-1)\mu - 2n(R+2a-1)(n+2R-1) - R^2 + (1-2a)R + a)], \end{aligned}$$

where $R = -x - a - 2$.

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